

Noncommutative Local Systems

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Gelfand - Naïmark theorem supplies a one to one correspondence between commutative C^* -algebras and locally compact Hausdorff spaces. So any noncommutative C^* -algebra can be regarded as a generalization of a topological space. Generalizations of several topological invariants may be defined by algebraic methods. For example Serre Swan theorem [16] states that complex topological K -theory coincides with K -theory of C^* -algebras. This article is concerned with generalization of local systems. The classical construction of local system implies an existence of a path groupoid. However the noncommutative geometry does not contain this object. There is a construction of local system which uses covering projections. Otherwise a classical (commutative) notion of a covering projection has a noncommutative generalization. A generalization of noncommutative covering projections supplies a generalization of local systems.

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1 Motivation. Preliminaries

Local system examples arise geometrically from vector bundles with flat connections, and from topology by means of linear representations of the fundamental group. Generalization of local systems requires a generalization of a topological space given by the Gelfand-Naimark theorem [1] which states the correspondence between locally compact Hausdorff topological spaces and commutative C^* -algebras.

Theorem 1.1. [1] *Let A be a commutative C^* -algebra and let \mathcal{X} be the spectrum of A . There is the natural $*$ -isomorphism $\gamma : A \rightarrow C_0(\mathcal{X})$.*

So any (noncommutative) C^* -algebra may be regarded as a generalized (noncommutative) locally compact Hausdorff topological space. We would like to generalize a notion of a local system. A classical notion of local system uses a fundamental groupoid.

Theorem 1.2. [23] *For each topological space there is a category $\mathcal{P}(\mathcal{X})$ whose objects are points of \mathcal{X} , whose morphisms from x_0 to x_1 are the path classes with x_0 as origin and x_1 as end, and whose composite is the product of path classes.*

Definition 1.3. [23] *The category $\mathcal{P}(\mathcal{X})$ is called the category of path classes of \mathcal{X} or the fundamental groupoid.*

Definition 1.4. [23] *A local system on a space \mathcal{X} is a covariant functor from fundamental groupoid of \mathcal{X} to some category. For any category \mathcal{C} there is a category of local systems on \mathcal{X} with values in \mathcal{C} . Two local systems are said to be equivalent if they are equivalent objects in this category.*

Otherwise it is known that any connected groupoid is equivalent to a category with single object, i.e. a groupoid is equivalent to a group which is regarded as a category. Any groupoid can be decomposed into connected components, therefore any local system corresponds to representations of groups. It means that in case of linearly connected space \mathcal{X} local systems can be defined by representations of fundamental group $\pi_1(\mathcal{X})$. Otherwise there is an interrelationship between fundamental group and covering projections. This circumstance supplies a following definition 1.5 and a lemma 1.6 which do not explicitly uses a fundamental groupoid.

Definition 1.5. [6] Let $p : \mathcal{P} \rightarrow \mathcal{B}$ be a principal G -bundle. Suppose G acts on the left on a space \mathcal{F} , i.e. an action $G \times \mathcal{F} \rightarrow \mathcal{F}$ is given. Define the *Borel construction*

$$\mathcal{P} \times_G \mathcal{F}$$

to be the quotient space $\mathcal{P} \times \mathcal{F} / \approx$ where

$$(x, f) \approx (xg, g^{-1}f).$$

We next give one application of the Borel construction. Recall that a local coefficient system is a fiber bundle over B with fiber A and structure group G where A is a (discrete) abelian group and G acts via a homomorphism $G \rightarrow \text{Aut}(A)$.

Lemma 1.6. [6] Every local coefficient system over a path-connected (and semilocally simply connected) space B is of the form

$$\begin{array}{ccc} A & \longrightarrow & \tilde{\mathcal{B}} \times_{\pi_1(\mathcal{B})} A \\ & & \downarrow q \\ & & B \end{array}$$

i.e., is associated to the principal $\pi_1(\mathcal{B})$ -bundle given by the universal cover $\tilde{\mathcal{B}}$ of \mathcal{B} where the action is given by a homomorphism $\pi_1(\mathcal{B}) \rightarrow \text{Aut}(A)$.

In lemma 1.6 the \mathcal{B} is a topological space, the $\tilde{\mathcal{B}}$ means the universal covering space of \mathcal{B} , $\pi = \pi_1(\mathcal{B})$ is the fundamental group of the \mathcal{B} . The π group equals to group of covering transformations $G(\tilde{\mathcal{B}}, \mathcal{B})$ of the universal covering space. So above construction does not need fundamental groupoid, it uses a covering projection and a group of covering transformations. However noncommutative generalizations of these notions are developed in [11]. So local systems can be generalized. We may summarize several properties of the Gelfand - Naimark correspondence with the following dictionary.

TOPOLOGY	ALGEBRA
Locally compact space	C^* -algebra
Covering projection	Noncommutative covering projection
Group of covering transformations	Noncommutative group of covering transformations
Local system	?

This article assumes elementary knowledge of following subjects:

1. Set theory [10],
2. Category theory [23],
3. Algebraic topology [23],

4. C^* -algebras and operator theory [20],
5. Differential geometry [17],
6. Spectral triples and their connections [4, 5, 8, 25].

The terms "set", "family" and "collection" are synonyms. Following table contains used in this paper notations.

Symbol	Meaning
A^G	Algebra of G invariants, i.e. $A^G = \{a \in A \mid ga = a, \forall g \in G\}$
$\text{Aut}(A)$	Group $*$ - automorphisms of C^* algebra A
$B(H)$	Algebra of bounded operators on Hilbert space H
\mathbb{C} (resp. \mathbb{R})	Field of complex (resp. real) numbers
$C(\mathcal{X})$	C^* - algebra of continuous complex valued functions on topological space \mathcal{X}
$C_0(\mathcal{X})$	C^* - algebra of continuous complex valued functions on topological space \mathcal{X}
$G(\tilde{\mathcal{X}} \mathcal{X})$	Group of covering transformations of covering projection $\tilde{\mathcal{X}} \rightarrow \mathcal{X}$ [23]
H	Hilbert space
$M(A)$	A multiplier algebra of C^* -algebra A
$\mathcal{P}(\mathcal{X})$	Fundamental groupoid of a topological space \mathcal{X}
$U(H) \subset \mathcal{B}(H)$	Group of unitary operators on Hilbert space H
$U(A) \subset A$	Group of unitary operators of algebra A
$U(n) \subset GL(n, \mathbb{C})$	Unitary subgroup of general linear group
\mathbb{Z}	Ring of integers
\mathbb{Z}_m	Ring of integers modulo m
Ω	Natural contravariant functor from category of commutative C^* - algebras, to category of Hausdorff spaces

2 Noncommutative covering projections

In this section we recall the described in [11] construction of a noncommutative covering projection. Instead the expired "rigged space" notion we use the "Hilbert module" one.

2.1 Hermitian modules and functors

Definition 2.1. [22] Let B be a C^* -algebra. By a (left) *Hermitian B -module* we will mean the Hilbert space H of a non-degenerate $*$ -representation $A \rightarrow B(H)$. Denote by $\mathbf{Herm}(B)$ the category of Hermitian B -modules.

Let A, B be C^* -algebras. In this section we will study some general methods for construction of functors from $\mathbf{Herm}(B)$ to $\mathbf{Herm}(A)$.

Definition 2.2. [22] Let B be a C^* -algebra. By (right) *pre- B -Hilbert module* we mean a vector space, X , over complex numbers on which B acts by means of linear transformations in such a way that X is a right B -module (in algebraic sense), and on which there is defined a B -valued sesquilinear form \langle, \rangle_X conjugate linear in the first variable, such that

1. $\langle x, x \rangle_B \geq 0$
2. $(\langle x, y \rangle_X)^* = \langle y, x \rangle_X$
3. $\langle x, yb \rangle_X = \langle x, y \rangle_X b$.

2.3. It is easily seen that if we factor a pre- B -Hilbert module by subspace of the elements x for which $\langle x, x \rangle_X = 0$, the quotient becomes in a natural way a pre- B -Hilbert module having the additional property that inner product is definite, i.e. $\langle x, x \rangle_X > 0$ for any non-zero $x \in X$. On a pre- B -Hilbert module with definite inner product we can define a norm $\| \cdot \|$ by setting

$$\|x\| = \|\langle x, x \rangle_X\|^{1/2}. \quad (1)$$

From now we will always view a pre- B -Hilbert module with definite inner product as being equipped with this norm. The completion of X with this norm is easily seen to become again a pre- B -Hilbert module.

Definition 2.4. [22] Let B be a C^* -algebra. By a *Hilbert B -module* we will mean a pre- B -Hilbert module, X , satisfying the following conditions:

1. If $\langle x, x \rangle_X = 0$ then $x = 0$, for all $x \in X$
2. X is complete for the norm defined in (1).

Example 2.5. Let A be a C^* -algebra and a finite group acts on A , A^G is the algebra of G -invariants. Then A is a Hilbert A^G -module on which is defined following A^G -valued form

$$\langle x, y \rangle_A = \frac{1}{|G|} \sum_{g \in G} g(x^*y). \quad (2)$$

Since given by 2 sum is G -invariant we have $\langle x, y \rangle_A \in A^G$.

Viewing a Hilbert B -module as a generalization of an ordinary Hilbert space, we can define what we mean by bounded operators on a Hilbert B -module.

Definition 2.6. [22] Let X be a Hilbert B -module. By a *bounded operator* on X we mean a linear operator, T , from X to itself which satisfies following conditions:

1. for some constant k_T we have

$$\langle Tx, Tx \rangle_X \leq k_T \langle x, x \rangle_X, \quad \forall x \in X,$$

or, equivalently T is continuous with respect to the norm of X .

2. there is a continuous linear operator, T^* , on X such that

$$\langle Tx, y \rangle_X = \langle x, T^*y \rangle_X, \quad \forall x, y \in X.$$

It is easily seen that any bounded operator on a B -Hilbert module will automatically commute with the action of B on X (because it has an adjoint). We will denote by $\mathcal{L}(X)$ (or $\mathcal{L}_B(X)$ there is a chance of confusion) the set of all bounded operators on X . Then it is easily verified that with the operator norm $\mathcal{L}(X)$ is a C^* -algebra.

Definition 2.7. [20] If X is a Hilbert B -module then denote by $\theta_{\xi, \zeta} \in \mathcal{L}_B(X)$ such that

$$\theta_{\xi, \zeta}(\eta) = \zeta \langle \xi, \eta \rangle_X, \quad (\xi, \eta, \zeta \in X)$$

Norm closure of a generated by such endomorphisms ideal is said to be the *algebra of compact operators* which we denote by $\mathcal{K}(X)$. The $\mathcal{K}(X)$ is an ideal of $\mathcal{L}_B(X)$. Also we shall use following notation $\tilde{\zeta} \langle \zeta \rangle \stackrel{\text{def}}{=} \theta_{\tilde{\zeta}, \zeta}$.

Definition 2.8. [22] Let A and B be C^* -algebras. By a *Hilbert B - A -correspondence* we mean a Hilbert B -module, which is a left A -module by means of $*$ -homomorphism of A into $\mathcal{L}_B(X)$.

2.9. Let X be a Hilbert B - A -correspondence. If $V \in \mathbf{Herm}(B)$ then we can form the algebraic tensor product $X \otimes_{B_{\text{alg}}} V$, and equip it with an ordinary pre-inner-product which is defined on elementary tensors by

$$\langle x \otimes v, x' \otimes v' \rangle = \langle \langle x', x \rangle_B v, v' \rangle_V.$$

Completing the quotient $X \otimes_{B_{\text{alg}}} V$ by subspace of vectors of length zero, we obtain an ordinary Hilbert space, on which A acts (by $a(x \otimes v) = ax \otimes v$) to give a $*$ -representation of A . We will denote the corresponding Hermitian module by $X \otimes_B V$. The above construction defines a functor $X \otimes_B - : \mathbf{Herm}(B) \rightarrow \mathbf{Herm}(A)$ if for $V, W \in \mathbf{Herm}(B)$ and $f \in \text{Hom}_B(V, W)$ we define $f \otimes X \in \text{Hom}_A(V \otimes X, W \otimes X)$ on elementary tensors by $(f \otimes X)(x \otimes v) = x \otimes f(v)$. We can define action of B on $V \otimes X$ which is defined on elementary tensors by

$$b(x \otimes v) = (x \otimes bv) = xb \otimes v.$$

2.2 Galois correspondences

Definition 2.10. Let A be a C^* -algebra, G is a finite or countable group which acts on A . We say that $H \in \mathbf{Herm}(A)$ is a A - G *Hermitian module* if

1. Group G acts on H by unitary A -linear isomorphisms,
2. There is a subspace $H^G \subset H$ such that

$$H = \bigoplus_{g \in G} gH^G. \tag{3}$$

Let H, K be A - G Hermitian modules, a morphism $\phi : H \rightarrow K$ is said to be a A - G -morphism if $\phi(gx) = g\phi(x)$ for any $g \in G$. Denote by $\mathbf{Herm}(A)^G$ a category of A - G Hermitian modules and A - G -morphisms.

Remark 2.11. Condition 2 in the above definition is introduced because any topological covering projection $\tilde{\mathcal{X}} \rightarrow \mathcal{X}$ commutative C^* algebras $C_0(\tilde{\mathcal{X}})$, $C_0(\mathcal{X})$ satisfies it with respect to the group of covering transformations $G(\tilde{\mathcal{X}}|\mathcal{X})$.

Definition 2.12. Let H be A - G Hermitian module, $B \subset M(A)$ is sub- C^* -algebra such that $(ga)b = g(ab)$, $b(ga) = g(ba)$, for any $a \in A$, $b \in B$, $g \in G$. There is a functor $(-)^G : \mathbf{Herm}(A)^G \rightarrow \mathbf{Herm}(B)$ defined by following way

$$H \mapsto H^G. \quad (4)$$

This functor is said to be the *invariant functor*.

Definition 2.13. Let ${}_A X_B$ be a Hilbert B - A correspondence, G is finite or countable group such that

- G acts on A and X ,
- Action of G is equivariant, i.e. $g(a\xi) = (ga)(g\xi)$, and B invariant, i.e. $g(\xi b) = (g\xi)b$ for any $\xi \in X$, $b \in B$, $a \in A$, $g \in G$,
- Inner-product of G is equivariant, i.e. $\langle g\xi, g\xi \rangle_X = \langle \xi, \xi \rangle_X$ for any $\xi, \zeta \in X$, $g \in G$.

Then we say that ${}_A X_B$ is a *G -equivariant Hilbert B - A -correspondence*.

Let ${}_A X_B$ be a G -equivariant Hilbert B - A -correspondence. Then for any $H \in \mathbf{Herm}(B)$ there is an action of G on $X \otimes_B H$ such that

$$g(x \otimes \xi) = (x \otimes g\xi).$$

Definition 2.14. Let ${}_A X_B$ be a G -equivariant Hilbert B - A -correspondence. We say that ${}_A X_B$ is *G -Galois Hilbert B - A -correspondence* if it satisfies following conditions:

1. $X \otimes_B H$ is a A - G Hermitian module, for any $H \in \mathbf{Herm}(B)$,
2. A pair $(X \otimes_B -, (-)^G)$ such that

$$X \otimes_B - : \mathbf{Herm}(B) \rightarrow \mathbf{Herm}(A)^G,$$

$$(-)^G : \mathbf{Herm}(A)^G \rightarrow \mathbf{Herm}(B).$$

is a pair of inverse equivalence.

Following theorem is an analog of to theorems described in [19], [24].

Theorem 2.15. [11] Let A and \tilde{A} be C^* -algebras, ${}_{\tilde{A}}X_A$ be a G -equivariant Hilbert A - \tilde{A} -correspondence. Let I be a finite or countable set of indices, $\{e_i\}_{i \in I} \subset M(A)$, $\{\tilde{\xi}_i\}_{i \in I} \subset {}_{\tilde{A}}X_A$ such that

$$1. \quad 1_{M(A)} = \sum_{i \in I} e_i^* e_i, \quad (5)$$

$$2. \quad 1_{M(K(X))} = \sum_{g \in G} \sum_{i \in I} g \tilde{\xi}_i \langle g \tilde{\xi}_i, \quad (6)$$

$$3. \quad \langle \tilde{\xi}_i, \tilde{\xi}_i \rangle_X = e_i^* e_i, \quad (7)$$

$$4. \quad \langle g \tilde{\xi}_i, \tilde{\xi}_i \rangle_X = 0, \text{ for any nontrivial } g \in G. \quad (8)$$

Then ${}_{\tilde{A}}X_A$ is a G -Galois Hilbert A - \tilde{A} -correspondence.

Definition 2.16. Consider a situation from the theorem 2.15. Let us consider two specific cases

1. $e_i \in A$ for any $i \in I$,
2. $\exists i \in I$ $e_i \notin A$.

Norm completion of the generated by operators

$$g \tilde{\xi}_i^* \langle g \tilde{\xi}_i a; g \in G, i \in I, \begin{cases} a \in M(A), & \text{in case 1,} \\ a \in A, & \text{in case 2.} \end{cases}$$

algebra is said to be the *subordinated to $\{\tilde{\xi}_i\}_{i \in I}$ algebra*. If \tilde{A} is the subordinated to $\{\tilde{\xi}_i\}_{i \in I}$ then

1. G acts on \tilde{A} by following way

$$g (g' \tilde{\xi}_i \langle g' \tilde{\xi}_i a) = g g' \tilde{\xi}_i \langle g g' \tilde{\xi}_i a; a \in M(A).$$

2. X is a left A module, moreover ${}_{\tilde{A}}X_A$ is a G -Galois Hilbert A - \tilde{A} -correspondence.
3. There is a natural G -equivariant $*$ -homomorphism $\varphi : A \rightarrow M(\tilde{A})$, φ is equivariant, i.e.

$$\varphi(a)(g \tilde{a}) = g \varphi(a)(\tilde{a}); a \in A, \tilde{a} \in \tilde{A}. \quad (9)$$

A quadruple $(A, \tilde{A}, {}_{\tilde{A}}X_A, G)$ is said to be a *Galois quadruple*. The group G is said to be a *group Galois transformations* which shall be denoted by $G(\tilde{A} | A) = G$.

Remark 2.17. Henceforth subordinated algebras only are regarded as noncommutative generalizations of covering projections.

Definition 2.18. If G is finite then bimodule ${}_{\tilde{A}}X_A$ can be replaced with ${}_{\tilde{A}}\tilde{A}_A$ where product $\langle , \rangle_{\tilde{A}}$ is given by (2). In this case a Galois quadruple $(A, \tilde{A}, {}_{\tilde{A}}X_A, G) = (A, \tilde{A}, {}_{\tilde{A}}A_A, G)$ can be replaced with a *Galois triple* (A, \tilde{A}, G) .

2.3 Infinite noncommutative covering projections

In case of commutative C^* -algebras definition 2.14 supplies algebraic formulation of infinite covering projections of topological spaces. However I think that above definition is not a quite good analogue of noncommutative covering projections. Noncommutative algebras contains inner automorphisms. Inner automorphisms are rather gauge transformations [9] than geometrical ones. So I think that inner automorphisms should be excluded. Importance of outer automorphisms was noted by Miyashita [18, 19]. It is reasonable to take into account outer automorphisms only. I have set more strong condition.

Definition 2.19. [21] Let A be C^* -algebra. A $*$ -automorphism α is said to be *generalized inner* if it is given by conjugating with unitaries from multiplier algebra $M(A)$.

Definition 2.20. [21] Let A be C^* -algebra. A $*$ -automorphism α is said to be *partly inner* if its restriction to some non-zero α -invariant two-sided ideal is generalized inner. We call automorphism *purely outer* if it is not partly inner.

Instead definitions 2.19, 2.20 following definitions are being used.

Definition 2.21. Let $\alpha \in \text{Aut}(A)$ be an automorphism. A representation $\rho : A \rightarrow B(H)$ is said to be α -invariant if a representation ρ_α given by

$$\rho_\alpha(a) = \rho(\alpha(a))$$

is unitary equivalent to ρ .

Definition 2.22. Automorphism $\alpha \in \text{Aut}(A)$ is said to be *strictly outer* if for any α -invariant representation $\rho : A \rightarrow B(H)$, automorphism ρ_α is not a generalized inner automorphism.

Definition 2.23. A Galois quadruple $(A, \tilde{A}, {}_{\tilde{A}}X_A, G)$ (resp. a triple (A, \tilde{A}, G)) with countable (resp. finite) G is said to be a *noncommutative infinite (resp. finite) covering projection* if action of G on \tilde{A} is strictly outer.

3 Noncommutative generalization of local systems

Definition 3.1. Let A be a C^* -algebra, and let \mathcal{C} be a category. A *noncommutative local system* contains following ingredients:

1. A noncommutative covering projection $(A, \tilde{A}, \tilde{X}_A, G)$ (or (A, \tilde{A}, G)),
2. A covariant functor $F : G \rightarrow \mathcal{C}$,

where G is regarded as a category with a single object e , which is the unity of G . Indeed a local system is a group homomorphism $G \rightarrow \text{Aut}(F(e))$.

Example 3.2. If \mathcal{X} is a linearly connected space then there is the equivalence of categories $\mathcal{P}(\mathcal{X}) \approx \pi_1(\mathcal{X})$. Let $F : \mathcal{P}(\mathcal{X}) \rightarrow \mathcal{C}$ is a local system then there is an object A in \mathcal{C} such that F is uniquely defined by a group homomorphism $f : \pi_1(\mathcal{X}) \rightarrow \text{Aut}(A)$. Let $G = \pi_1(\mathcal{X})/\ker f$ be a factor group and let $\tilde{\mathcal{X}} \rightarrow \mathcal{X}$ be a covering projection such that $G(\tilde{\mathcal{X}}|\mathcal{X}) \approx G$. Then there is a natural group homomorphism $G \rightarrow \text{Aut}(A)$ which can be regarded as covariant functor $G \rightarrow \mathcal{C}$. If \mathcal{X} is locally compact and Hausdorff then from [11] it follows that there is a noncommutative covering projection $(C_0(\mathcal{X}), C_0(\tilde{\mathcal{X}}), {}_{C_0(\mathcal{X})}X_{C_0(\tilde{\mathcal{X}})}, G)$. So a noncommutative local system is a generalization of a commutative one.

4 Noncommutative bundles with flat connections

4.1 Cotensor products

4.1. Cotensor products associated with Hopf algebras. Let H be a Hopf algebra over a commutative ring k , with bijective antipode S . We use the Sweedler notation [15] for the comultiplication on H : $\Delta(h) = h_{(1)} \otimes h_{(2)}$. \mathcal{M}^H (respectively ${}^H\mathcal{M}$) is the category of right (respectively left) H -comodules. For a right H -coaction ρ (respectively a left H -coaction λ) on a k -module M , we denote

$$\rho(m) = m_{[0]} \otimes m_{[1]} \quad \text{and} \quad \lambda(m) = m_{[-1]} \otimes m_{[0]}.$$

Let M be a right H -comodule, and N a left H -comodule. The cotensor product $M \square_H N$ is the k -module

$$M \square_H N = \left\{ \sum_i m_i \otimes n_i \in M \otimes N \mid \sum_i \rho(m_i) \otimes n_i = \sum_i m_i \otimes \lambda(n_i) \right\}. \quad (10)$$

If H is cocommutative, then $M \square_H N$ is also a right (or left) H -comodule.

4.2. Cotensor products associated with groups. Let G be a finite group. A set $H = \text{Map}(G, \mathbb{C})$ has a natural structure of commutative Hopf algebra (See [13]). Addition (resp. multiplication) on H is pointwise addition (resp. pointwise multiplication). Let $\delta_g \in H$, ($g \in G$) be such that

$$\delta_g(g') \begin{cases} 1 & g' = g \\ 0 & g' \neq g \end{cases} \quad (11)$$

Comultiplication $\Delta : H \rightarrow H \otimes H$ is induced by group multiplication

$$\Delta f(g) = \sum_{g_1 g_2 = g} f(g_1) \otimes f(g_2); \quad \forall f \in \text{Map}(G, \mathbb{C}), \quad \forall g \in G.$$

i.e.

$$\Delta\delta_g = \sum_{g_1 g_2 = g} \delta_{g_1} \otimes \delta_{g_2}; \quad \forall g \in G,$$

Let M (resp. N) be a linear space with right (resp. left) action of G then

$$M \square_{\text{Map}(G, \mathbb{C})} N = \left\{ \sum_i m_i \otimes n_i \in M \otimes N \mid \sum_i m_i g \otimes n_i = \sum_i m_i \otimes g n_i; \quad \forall g \in G \right\}. \quad (12)$$

Henceforth we denote by $M \square_G N$ a cotensor product $M \square_{\text{Map}(G, \mathbb{C})} N$.

4.2 Bundles with flat connections in differential geometry

I follow to [17] in explanation of the differential geometry and flat bundles.

Proposition 4.3. (*Proposition 5.9 [17]*)

1. Given a connected manifold M there is a unique (unique up to isomorphism) universal covering manifold, which will be denoted by \tilde{M} .
2. The universal covering manifold \tilde{M} is a principal fibre bundle over M with group $\pi_1(M)$ and projection $p : \tilde{M} \rightarrow M$, where $\pi_1(M)$ is the first homotopy group of M .
3. The isomorphism classes of covering spaces over M are in 1:1 correspondence with the conjugate classes of subgroups of $\pi_1(M)$. The correspondence is given as follows. To each subgroup H of $\pi_1(M)$, we associate $E = \tilde{M}/H$. Then the covering manifold E corresponding to H is a fibre bundle over M with fibre $\pi_1(M)/H$ associated with the principal bundle $\tilde{M}(M, \pi_1(M))$. If H is a normal subgroup of $\pi_1(M)$, $E = \tilde{M}/H$ is a principal fibre bundle with group $\pi_1(M)/H$ and is called a regular covering manifold of M .

Let Γ be a flat connection $P(M, G)$, where M is connected and paracompact. Let $u_0 \in P$; $M^* = P(u_0)$, the holonomy bundle through u_0 ; M^* is a principal fibre bundle over M whose structure group is the holonomy group $\Phi(u_0)$. In [17] is explained that $\Phi(u_0)$ is discrete, and since M^* is connected, M^* is a covering space of M . Set $x_0 = \pi(u_0) \in M$. Every closed curve of M starting from x_0 defines, by means of the parallel displacement along it, an element of $\Phi(u_0)$. In [17] it is explained that the same element of the first homotopy group $\pi_1(M, x_0)$ give rise to the same element of $\Phi(u_0)$. Thus we obtain a homomorphism of $\pi_1(M, x_0)$ onto $\Phi(u_0)$. Let N be a normal subgroup of $\Phi(u_0)$ and set $M' = M^*/N$. Then M' is principal fibre bundle over M with structure group $\Phi(u_0)/N$. In particular M' is a covering space of M . Let $P'(M', G)$ be the principal fibre bundle induced by covering projection $M' \rightarrow M$. There is a natural homomorphism $f : P' \rightarrow P$ [17].

Proposition 4.4. (*Proposition 9.3 [17]*) There exists a unique connection Γ' in $P'(M', G)$ which is mapped into Γ by homomorphism $f : P' \rightarrow P$. The connection Γ' is flat. If u'_0 is a point of P' such that $f(u'_0) = u_0$, then the holonomy group $\Phi(u'_0)$ of Γ' with reference point u'_0 is isomorphically mapped onto N by f .

4.5. Construction of flat connections Let M be a manifold. Proposition 4.4 supplies construction of flat bundle $P(M, G)$ which imply following ingredients:

1. A covering projection $M' \rightarrow M$.
2. A principal bundle $P'(M', G)$ with a flat connection Γ .

4.6. Associated vector bundle. A principal bundle $P(M, G)$ and a flat connection Γ are given by these ingredients. If G acts on \mathbb{C}^n then there is an associated with $P(M, G)$ vector fibre \mathcal{F} bundle with a standard fibre \mathbb{C}^n . A space F of continuous sections of \mathcal{F} is a finitely generated projective $C(M)$ -module. See [17].

4.7. Canonical flat connection and flat bundles. There is a specific case of flat principal bundle such that $P' = M' \times G$ and Γ is a canonical flat connection [17]. In this case the existence of $P(M, G)$ depends only on $\pi_1(M)$ and does not depend on differential structure of M .

4.8. Local systems and K-theory. If $R(G)$ is the group representation ring and $R_0(G)$ is a subgroup of zero virtual dimension then there is a natural homomorphism $R_0(G) \rightarrow K^0(M)$ described in [7, 26].

4.3 Topological noncommutative bundles with flat connections

There are noncommutative generalizations of described in 4.2 constructions. According to Serre Swan theorem [16] any vector bundle over space \mathcal{X} corresponds to a projective $C_0(\mathcal{X})$ module.

Definition 4.9. Let (A, \tilde{A}, G) be a finite noncommutative covering projection. According to definition 3.1 any group homomorphism $G \rightarrow U(n)$ is a local system. There is a natural linear action of G on \mathbb{C}^n , and $\tilde{A} \square_G \mathbb{C}^n$ is a left A -module which is said to be a *topological noncommutative bundle with flat connection*.

Lemma 4.10. Let (A, \tilde{A}, G) be a finite noncommutative covering projection, and let $P = \tilde{A} \square_G \mathbb{C}^n$ be a topological noncommutative bundle with flat connection. Then P is a finitely generated projective left and right A -module.

Proof. According to definition \tilde{A} is a left finitely generated projective A -module. A left A -module $\tilde{A} \otimes_{\mathbb{C}} \mathbb{C}^n$ is also finitely generated and projective because $\tilde{A} \otimes_{\mathbb{C}} \mathbb{C}^n \approx \tilde{A}^n$. There is a projection $p : \tilde{A} \otimes_{\mathbb{C}} \mathbb{C}^n \rightarrow \tilde{A} \otimes_{\mathbb{C}} \mathbb{C}^n$ given by:

$$p(a \otimes x) = \frac{1}{|G|} \sum_{g \in G} ag \otimes g^{-1}x.$$

The image of p is P , therefore P is projective left A -module. Similarly we can prove that P is a finitely generated projective right A -module \square

Example 4.11. Let M be a differentiable manifold $M' \rightarrow M$ is a covering projection $P' = M' \times U(n)$ is a principal bundle with a canonical flat connection Γ' . So there are all ingredients of 4.5. So we have a principal bundle $P(M, U(n))$ with a flat connection Γ . There is a noncommutative covering projection $(C(M), C(M'), {}_{C(M')}X_{C(M)}, G)$. Let \mathcal{F} (resp. \mathcal{F}') be a vector bundle associated with $P(M, U(n))$ (resp. $P(M', U(n))$), and let F (resp. F') be a projective finitely generated $C(M)$ (resp. $C(M')$) module which corresponds to \mathcal{F} (resp. \mathcal{F}'). Then we have $F = C(M') \square_G F'$, i.e. F is a topological flat bundle.

Remark 4.12. Since existence of $P(M, U(n))$ depend on topology of M only we use a notion "topological noncommutative bundle with flat connection" is used for its noncommutative generalization.

Example 4.13. Let A_θ be a noncommutative torus $(A_\theta, A_{\theta'}, \mathbb{Z}_m \times \mathbb{Z}_n)$ a Galois triple described in [11]. Any group homomorphism $\mathbb{Z}_m \times \mathbb{Z}_n \rightarrow U(1)$ induces a topological noncommutative flat bundle.

4.4 General noncommutative bundles with flat connections

4.14. A vector fibre bundle with a flat connection is not necessary a topological bundle with flat connection, since proposition 4.2 and construction 4.5 does not require it. However general case of 4.2 and construction 4.5 have a noncommutative analogue. The analogue requires a noncommutative generalization of differentiable manifolds with flat connections. Generalization of a spin manifold is a spectral triple [4, 5, 8, 25]. First of all we generalize the proposition 4.3.

Suppose that there is a spectral triple (\mathcal{B}, H, D) such that

- $\mathcal{B} \subset B$ is a pre- C^* -algebra which is a dense subalgebra in B .
- there is a faithful representation $B \rightarrow B(H)$.

Let (B, A, G) be a finite noncommutative covering projection. According to 8.2 of [11] there is the spectral triple $(\mathcal{A}, A \otimes_B H, \tilde{D})$ such that

- $\mathcal{A} \subset A$ is a pre- C^* -algebra which is a dense subalgebra of A .
- $\tilde{D}gh = g\tilde{D}h$, for any $g \in G, h \in \text{Dom } \tilde{D}$.

Let \mathcal{F} be a finite projective right \mathcal{B} -module with a flat connection $\nabla : \mathcal{F} \rightarrow \mathcal{F} \otimes_B \Omega^1(\mathcal{B})$. Let $\mathcal{E} = \mathcal{F} \otimes_B \mathcal{A}$ be a projective finitely generated \mathcal{A} -module and the action of G on \mathcal{E} is induced by the action of G on \mathcal{A} . According to [11] connection ∇ can be naturally lifted to $\tilde{\nabla} : \mathcal{E} \rightarrow \mathcal{E} \otimes_B \Omega^1(\mathcal{B})$. Let \mathcal{E}' be an isomorphic to \mathcal{E} as \mathcal{A} -module and there is an action of G on \mathcal{E}' such that

$$g(xa) = (gx)(ga); \forall x \in \mathcal{E}, \forall a \in \mathcal{A}, \forall g \in G. \quad (13)$$

Different actions of G give different \mathcal{B} -modules $\mathcal{F} = \mathcal{E} \square_G \mathcal{A}, \mathcal{F}' = \mathcal{E}' \square_G \mathcal{A}$. Both \mathcal{F} and \mathcal{F}' can be included into following sequences

$$\mathcal{F} \xrightarrow{i} \mathcal{E} \xrightarrow{p} \mathcal{F}, \quad (14)$$

$$\mathcal{F}' \xrightarrow{i'} \mathcal{E}' \xrightarrow{p'} \mathcal{F}'.$$

These sequences induce following

$$\begin{aligned} \mathcal{F} \otimes_{\mathcal{B}} \Omega^1(\mathcal{B}) &\xrightarrow{i \otimes \text{Id}_{\Omega^1(\mathcal{B})}} \mathcal{E} \otimes_{\mathcal{B}} \Omega^1(\mathcal{B}) \xrightarrow{p \otimes \text{Id}_{\Omega^1(\mathcal{B})}} \mathcal{F} \otimes_{\mathcal{B}} \Omega^1(\mathcal{B}), \\ \mathcal{F}' \otimes_{\mathcal{B}} \Omega^1(\mathcal{B}) &\xrightarrow{i' \otimes \text{Id}_{\Omega^1(\mathcal{B})}} \mathcal{E}' \otimes_{\mathcal{B}} \Omega^1(\mathcal{B}) \xrightarrow{p' \otimes \text{Id}_{\Omega^1(\mathcal{B})}} \mathcal{F}' \otimes_{\mathcal{B}} \Omega^1(\mathcal{B}), \end{aligned} \quad (15)$$

The connection ∇ is given by

$$\nabla p(x) = \left(p \otimes \text{Id}_{\Omega^1(\mathcal{B})} \right) \left(\widetilde{\nabla}(x) \right); x \in \mathcal{E}.$$

From (14) and (15) it follows that if $y \in \mathcal{F}$ then ∇y does not depend on $x \in \mathcal{E}$ such that $y = p(x)$. Similarly there is a flat connection $\nabla' : \mathcal{F}' \rightarrow \mathcal{F}' \otimes_{\mathcal{B}} \Omega^1(\mathcal{B})$ given by

$$\nabla' p'(x) = \left(p' \otimes \text{Id}_{\Omega^1(\mathcal{B})} \right) \left(\widetilde{\nabla}'(x) \right); x \in \mathcal{E}'.$$

Following table explains a correspondence between the proposition 4.3 and the above construction.

DIFFERENTIAL GEOMETRY	SPECTRAL TRIPLES
Manifold M	Spectral triple (\mathcal{B}, H, D)
The covering manifold E	Spectral triple $(\mathcal{A}, \mathcal{A} \otimes_{\mathcal{B}} H, \widetilde{D})$
A regular covering projection $E \rightarrow M$	A noncommutative covering projection (B, A, G)
Group of covering transformations $\pi_1(M)/H$	Group of noncommutative covering transformations G
A connection on vector fibre bundle $F \rightarrow M$	An operator $\nabla : \mathcal{F} \rightarrow \mathcal{F} \otimes_{\mathcal{B}} \Omega^1(\mathcal{B})$

Example 4.15. Let $(\mathcal{A}_\theta, H, D)$ be a spectral triple associated to a noncommutative torus A_θ generated by unitary elements $u, v \in A_\theta$. Let $\mathcal{F} = \mathcal{A}_\theta^4$ be a free module and let $e_1, \dots, e_4 \in \mathcal{F}$ be its generators. Let $\nabla : \mathcal{F} \rightarrow \mathcal{F} \otimes \Omega^1(\mathcal{A}_\theta)$ be a connection given by

$$\nabla e_1 = c_u e_2 \otimes du, \nabla e_2 = -c_u e_1 \otimes du, \nabla e_3 = c_v e_4 \otimes dv, \nabla e_4 = -c_v e_3 \otimes dv.$$

where $c_u, c_v \in \mathbb{R}$. According to [12] the connection ∇ is flat. Let $(A_\theta, A_{\theta'}, \mathbb{Z}_m \times \mathbb{Z}_n)$ a Galois triple from example 4.13. This data induces a spectral triple $(\mathcal{A}_{\theta'}, \mathcal{A}_{\theta'} \otimes_{\mathcal{A}_\theta} H, D)$. If $\mathcal{E} = \mathcal{F} \otimes_{\mathcal{A}_\theta} \mathcal{A}_{\theta'}$ then

$$\mathcal{E} \approx \mathcal{A}_{\theta'} \otimes \mathbb{C}^4 \approx \mathcal{A}_\theta \otimes \mathbb{C}^{4nm} \quad (16)$$

and there is a natural connection $\widetilde{\nabla} : \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{A}_\theta} \Omega^1(\mathcal{A}_\theta)$. Let $\rho : \mathbb{Z}_m \times \mathbb{Z}_n \rightarrow U(4)$ be a nontrivial representation. There is an action of $\mathbb{Z}_m \times \mathbb{Z}_n$ on $\mathcal{E}' = \mathcal{A}_{\theta'} \otimes \mathbb{C}^4$ given by

$$g(a \otimes x) = ga \otimes \rho(g)x; a \in \mathcal{A}_{\theta'}, x \in \mathbb{C}^4.$$

which satisfies (13). Then $\mathcal{F}' = \mathcal{A}_{\theta'} \square_{\mathbb{Z}_m \times \mathbb{Z}_n} \mathcal{E}'$ is a finitely generated A_θ module with a connection $\nabla' : \mathcal{F}' \rightarrow \mathcal{F} \otimes \Omega^1(\mathcal{A}_\theta)$ given by the construction 4.14.

4.5 Noncommutative bundles with flat connections and K -theory

A homomorphism $R_0(G) \rightarrow K^0(M)$ from 4.8 can be generalized. Let (A, \tilde{A}, G) be a finite noncommutative covering projection and $\rho : G \rightarrow U(n)$ is a representation, $\text{triv}_n : G \rightarrow U(n)$ is the trivial representation. Suppose that an action of G on \mathbb{C}^n is given by ρ . Then a homomorphism $R_0(G) \rightarrow K(A)$ is given by

$$[\rho] - [\text{triv}_n] \mapsto [\tilde{A} \square_G \mathbb{C}^n] - [A^n].$$

5 Noncommutative generalization of Borel construction

5.1. There is a noncommutative generalization of the Borel construction 1.5

Definition 5.2. Let A, B be C^* -algebras, let G be a group which acts on both A and B . Let $A \otimes_{\mathbb{C}} B$ is any tensor product such that $A \otimes_{\mathbb{C}} B$ is a C^* -algebra. The norm closure of generated by

$$C = \left\{ \sum_i a_i \otimes b_i \in A \otimes_{\mathbb{C}} B \mid \sum_i a_i g \otimes b_i = \sum_i a_i \otimes g b_i; \forall g \in G \right\}.$$

subalgebra is said to be a *cotensor product of C^* -algebras*. Denote by $A \square_G B$ the cotensor product.

Remark 5.3. We do not fix a type of a tensor product because different applications can use different tensor products (See [2]).

Example 5.4. Let \mathcal{X}, \mathcal{Y} be locally compact Hausdorff spaces and let G be a finite or countable group which acts on both \mathcal{X} and \mathcal{Y} . Suppose that action on \mathcal{X} (resp. \mathcal{Y}) is right (resp. left). Then there is natural right (resp. left) action of on $C_0(\mathcal{X})$ (resp. $C_0(\mathcal{Y})$). From [2] it follows that the minimal and the maximal norm on $C_0(\mathcal{X}) \otimes_{\mathbb{C}} C_0(\mathcal{Y})$ coincide. It is well known that $C_0(\mathcal{X} \times \mathcal{Y}) \approx C_0(\mathcal{X}) \otimes_{\mathbb{C}} C_0(\mathcal{Y})$. Let $\mathcal{Z} = \mathcal{X} \times \mathcal{Y} / \approx$ where \approx is given by

$$(xg, y) \approx (x, g^{-1}y).$$

It is clear that $C_0(\mathcal{Z}) \approx C_0(\mathcal{X}) \square_G C_0(\mathcal{Y})$.

Definition 5.5. Let $(A, \tilde{A}, \tilde{A} X_A, G)$ be a Galois quadruple such that there is right action of G of \tilde{A} and left action of G on C^* -algebra B . A cotensor product $\tilde{A} \square_G B$ is said to be a *noncommutative Borel construction*.

Example 5.6. Let $p : \tilde{\mathcal{B}} \rightarrow \mathcal{B}$ be a topological normal covering projection of locally compact topological spaces, and $G = G(\tilde{\mathcal{B}}|\mathcal{B})$ is a group of covering transformations. Then p is a principal $G(\tilde{\mathcal{B}}|\mathcal{B})$ -bundle. Let \mathcal{F} be a locally compact topological space with action of G on it. Then there is a natural isomorphism with the C^* -algebra of a topological Borel construction

$$C_0(\tilde{\mathcal{B}} \times_G \mathcal{F}) \approx C_0(\tilde{\mathcal{B}}) \square_G C_0(\mathcal{F}).$$

References

- [1] W. Arveson. *An Invitation to C^* -Algebras*, Springer-Verlag. ISBN 0-387-90176-0, 1981.
- [2] Franka Miriam Brückler. *Tensor products of C^* -algebras, operator spaces and Hilbert C^* -modules*. Mathematical Communications 4(1999), 1999.
- [3] S. Caenepeel, S. Crivei, A. Marcus, M. Takeuchi. *Morita equivalences induced by bimodules over Hopf-Galois extensions*. arXiv:math/0608572, 2007.
- [4] Alain Connes. *C^* -algebras and differential geometry*. arXiv:hep-th/0101093, 2001.
- [5] Alain Connes. *Noncommutative Geometry*, Academic Press, San Diego, CA, 661 p., ISBN 0-12-185860-X, 1994.
- [6] James F. Davis. Paul Kirk *Lecture Notes in Algebraic Topology*. Department of Mathematics, Indiana University, Bloomington, IN 47405, 2001.
- [7] P.B. Gilkey. *The eta invariant and the K-theory of odd dimensional spherical space forms*. Inventiones mathematicae, Springer-Verlag, 1984.
- [8] José M. Gracia-Bondia, Joseph C. Varilly, Hector Figueroa, *Elements of Noncommutative Geometry*, Springer, 2001.
- [9] David J. Gross. *Gauge Theory-Past, Present, and Future?* Joseph Henry Luborutoties, Ainceton University, Princeton, NJ 08544, USA. (Received November 3,1992).
- [10] Paul R. Halmos *Naïve Set Theory*. D. Van Nostrand Company, Inc., Prineston, N.J., 1960.
- [11] Petr Ivankov. *Infinite Noncommutative Covering Projections*. arXiv:1405.1859, 2014.
- [12] Petr Ivankov. *Noncommutative Generalization of Wilson Lines*. arXiv:1408.4101, 2014.
- [13] *Lecture notes on noncommutative geometry and quantum groups*, Edited by Piotr M. Hajac.
- [14] Evgenios T.A. Kakariadis, Elias G. Katsoulis, *Operator algebras and C^* -correspondences: A survey*. arXiv:1210.6067, 2012.
- [15] Gizem Karaali *On Hopf Algebras and Their Generalizations*, arXiv:math/0703441, 2007.
- [16] M. Karoubi. *K-theory, An Introduction*. Springer-Verlag 1978.
- [17] S. Kobayashi, K. Nomizu. *Foundations of Differential Geometry*. Volume 1. Interscience publishers a division of John Willey & Sons, New York - London. 1963.
- [18] Yôichi Miyashita, *Finite outer Galois theory of noncommutative rings*. Department of Mathematics, Hokkaido, University, 1966.
- [19] Yôichi Miyashita, *Locally finite outer Galois theory*. Department of Mathematics, Hokkaido, University, 1967.

- [20] Pedersen G.K. *C*-algebras and their automorphism groups*. London ; New York : Academic Press, 1979.
- [21] Marc A. Reiffel, *Actions of Finite Groups on C* - Algebras*. Department of Mathematics University of California Berkeley. Cal. 94720 U.S.A. 1980.
- [22] Marc A. Reiffel, *Morita equivalence for C*-algebras and W*-algebras* , Journal of Pure and Applied Algebra 5 (1974), 51-96. 1974.
- [23] E.H. Spanier. *Algebraic Topology*. McGraw-Hill. New York 1966.
- [24] Takeuchi, Yasuji *Infinite outer Galois theory of non commutative rings* Osaka J. Math. Volume 3, Number 2, 1966.
- [25] J.C. Várilly. *An Introduction to Noncommutative Geometry*. EMS 2006.
- [26] Wolf, J. *Spaces of constant curvature*. New York: McGraw-Hill, 1967.